APPROXIMATE ANALYSIS OF A MULTICOMPONENT REACTING WAKE

L. I. Skurin and E. K. Steklyannikova

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An integral method is proposed for analysis of a multicomponent reacting far wake behind a body. The results of the analysis, obtained by using the method proposed, are compared with exact numerical data.

It is assumed that the flow in the far axisymmetric wake behind a body is described by a system of multicomponent boundary-layer equations:

$$\frac{\partial \rho v_x r}{\partial x} + \frac{\partial \rho v_r r}{\partial r} = 0,$$

$$\rho v_x \frac{\partial v_x}{\partial x} + \rho v_r \frac{\partial v_x}{\partial r} = -\frac{dp}{dx} + \frac{1}{r}\left[r(\mu_T + \mu)\frac{\partial v_x}{\partial r}\right], \tag{1}$$

$$\rho v_x \frac{\partial H}{\partial x} + \rho v_r \frac{\partial H}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left\{r\left(\frac{\mu_T}{\mathrm{Pr}_T} + \frac{\mu}{\mathrm{Pr}}\right)\frac{\partial H}{\partial r} + r\left[\mu_T\left(1 - \frac{1}{\mathrm{Pr}_T}\right) + \mu\left(1 - \frac{1}{\mathrm{Pr}}\right)\right] v_x \frac{\partial v_x}{\partial r}\right. \tag{2}$$

$$\left. + r\sum_{i=1}^{N}\left[\left(\frac{1}{\mathrm{Sc}_{Ti}} - \frac{1}{\mathrm{Pr}_T}\right)\mu_T \frac{\partial \xi_i}{\partial r} - \left(J_{r_i} + \frac{\mu}{\mathrm{Pr}}\frac{\partial \xi_i}{\partial r}\right)\right] h_i\right\}, \tag{3}$$

$$\rho v_x \frac{\partial \xi_i}{\partial x} + \rho v_r \frac{\partial \xi_i}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\mu_T}{\mathrm{Sc}_T}\frac{\partial \xi_i}{\partial r} - J_{r_i}\right)\right] + w_i, \tag{4}$$

$$i = 1, 2, \ldots, N,$$

$$\frac{\rho}{\rho_e} = T_e \sum_{i=1}^{N}\frac{\xi_{ie}}{M_i} \Big/ \left(T\sum_{i=1}^{N}\frac{\xi_i}{M_i}\right),$$

$$H = \frac{1}{(\varkappa - 1)\mathrm{M}_\infty^2}\sum_{i=1}^{N} h_i \xi_i + \frac{v_x^2}{2} \tag{5}$$

under the boundary conditions

$$r = 0, \quad v_r = \partial v_x/\partial r = \partial T/\partial r = \partial \xi_i/\partial r,$$

$$r \to \infty, \quad v_x \to v_e, \quad T \to T_e, \quad \xi_i \to \xi_{ie},$$

$$x = x_H, \quad v_x = v_H(r), \quad T = T_H(r), \quad \xi_i = \xi_{iH}(r). \tag{6}$$

All the notation here is dimensionless in conformity with the list of notations.

Let us use the integral relations method for an approximate solution of the problem (1)-(6). Let us set

$$\frac{v_e - v_x}{v_e u_0} = \exp\left[-\left(\frac{\eta}{\Delta}\right)^2 \ln 2\right] \equiv y, \quad u_0 \equiv \frac{v_e - v_0}{v_e}, \tag{7}$$

$$t/t_0 = (1 - a)\, y^{\mathrm{Pr}^*} + a y^{2\mathrm{Pr}^*}, \quad \mathrm{Pr}^* = \mathrm{const}, \quad t = \frac{T}{T_e} - 1, \tag{8}$$

$$\chi_i = (\chi_{i0} - l_i)\, y^{\mathrm{Sc}_i^*} + l_i y^{2\mathrm{Sc}_i^*}, \quad \mathrm{Sc}_i^* = \mathrm{const}, \quad \chi_i = \xi_i - \xi_{ie}, \tag{9}$$

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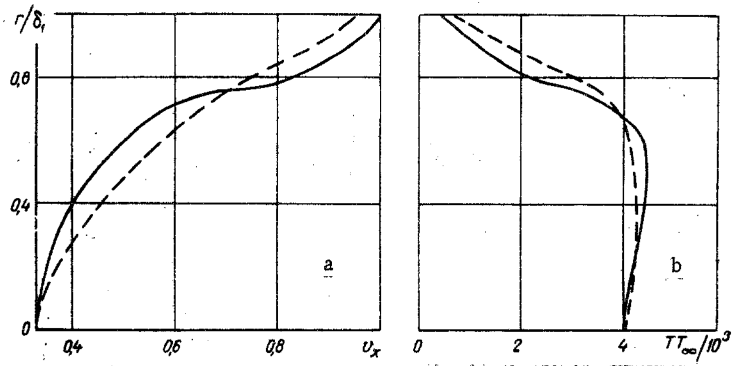

Fig. 1. Initial velocity (a) and temperature (b) profiles: dashes reflect data from this paper, while solid lines are from [1].

where

$$\eta^2 = 2\int_0^r (\rho/\rho_e)\, r\,dr, \quad \Delta^2 = 2\int_0^\delta (\rho/\rho_e)\, r\,dr, \quad v_e - v_x(\delta) = \frac{v_e u_0}{2}. \tag{10}$$

Let us note that the representations (7)-(10) satisfy all the conditions imposed on the corresponding functions by the relationships (6) for r = 0 and r → ∞.

From the equation of state and (8) and (9) there follows that

$$\bar\rho \equiv \rho/\rho_e = \{1 + t_0[(1-a)\,y^{\mathrm{Pr}^*} + a y^{2\mathrm{Pr}^*}]\}^{-1}\left\{1 - \left(\sum_{i=1}^N \frac{\xi_{ie}}{M_i}\right)^{-1} \sum_{i=1}^N \frac{1}{M_i}\left[(\chi_{i0} - l_i)\, y^{\mathrm{Sc}_i^*} + l_i y^{2\mathrm{Sc}_i^*}\right]\right\}^{-1}. \tag{11}$$

By using (7) and (11) a connection between the physical coordinate r and the transformed coordinate η (or y) can be obtained in analytical form from the first relationship in (10).

To determine the unknown functions of the longitudinal coordinate u_0, Δ, t_0, a, χ_{i0}, l_i let us require that integral relations be satisfied for (2)-(4), as well as that (2) and (4) and the heat influx equation

$$\rho c_p\left(v_x \frac{\partial T}{\partial x} + v_r \frac{\partial T}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[r c_p\left(\frac{\mu_T}{\mathrm{Pr}_T} + \frac{\mu}{\mathrm{Pr}}\right)\frac{\partial T}{\partial r}\right]$$
$$+ (\varkappa - 1)\,M_\infty^2\left[v_x \frac{dp}{dx} + (\mu_T + \mu)\left(\frac{\partial v_x}{\partial r}\right)^2\right] - \sum_{i=1}^N h_i w_i$$
$$+ \sum_{i=1}^N\left(\frac{\mu_T}{\mathrm{Sc}_{Ti}}\frac{\partial \xi_i}{\partial r} - J_{ri}\right)\frac{\partial h_i}{\partial r}, \quad c_p \equiv \sum_{i=1}^N \frac{\partial h_i}{\partial T}\xi_i$$

be satisfied for r = 0.

The axial equations and integral relations for the equations of motion, energy, and concentration conservation are, respectively:

$$\frac{du_0}{dx} = -4\ln 2\,\frac{(\bar\mu_T + \bar\mu_0)\,u_0}{\Delta^2(1-u_0)} - \left[\frac{1}{\bar\rho_0(1-u_0)} - 1 + u_0\right]\frac{d\ln v_e}{dx}, \tag{12}$$

$$\frac{d\Delta}{dx} = -\frac{\Delta(1-u_0)}{u_0(2-u_0)}\frac{du_0}{dx} - \frac{\Delta}{2}\frac{d\ln \rho_e v_e^2}{dx} - \frac{\Delta(u_0 + A)}{u_0(2-u_0)}\frac{d\ln v_e}{dx}, \tag{13}$$

$$\frac{dt_0}{dx} = -4\mathrm{Pr}^*\ln 2\,\frac{1+a}{1-u_0}\,\frac{t_0}{\Delta^2}\left(\frac{\bar\mu_T}{\mathrm{Pr}_T} + \frac{\bar\mu_0}{\mathrm{Pr}}\right)$$
$$- \frac{1}{T_e c_{p0}\bar\rho_0(1-u_0)}\sum_{i=1}^N \bar w_{i0} h_{i0} - (1+t_0)\frac{d\ln T_e}{dx} - \frac{(\varkappa-1)\,M_\infty^2 v_e}{T_e c_{p0}\bar\rho_0}\frac{dv_e}{dx}, \tag{14}$$

$$\int_0^1 \sum_{i=1}^N (h_{ie}\xi_{ie} - h_i\xi_i)(1-u_0 y)\frac{dy}{y} = (\varkappa-1)\,M_\infty^2 v_e^2\left[\frac{2\ln 2}{\rho_e v_e^3\Delta^2}\cdot\frac{c_T}{16} - u_0\left(1 - \frac{3}{4}u_0 + \frac{u_0^2}{6}\right)\right], \tag{15}$$

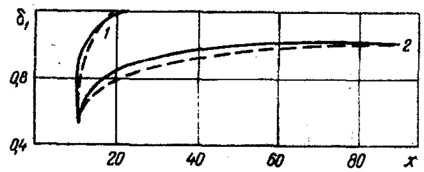

Fig. 2. Wake radius.

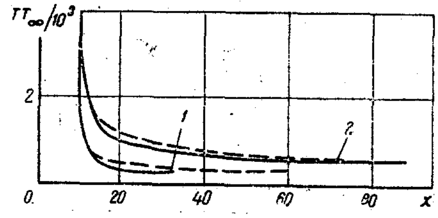

Fig. 3. Axial temperature.

$$\frac{d\xi_{i0}}{dx} = -4\,\mathrm{Sc}_i^*\ln 2\,\frac{\chi_{i0}+l_i}{\Delta^2(1-u_0)}\left(\frac{\bar{\mu}_T}{\mathrm{Sc}_{Ti}}+\frac{\bar{\mu}_0}{\mathrm{Sc}_i}\right)+\frac{\bar{w}_{i0}}{\bar{\rho}_0(1-u_0)},\quad i=1,\,2,\,\ldots,\,N-2, \tag{16}$$

$$\frac{d\alpha_i}{dx} = \frac{\Delta^2\rho_e v_e}{2\ln 2}J_i,\quad i=1,\,2,\,\ldots,\,N-2. \tag{17}$$

where

$$\bar{\mu}\equiv\frac{\mu}{\rho_e v_e},\quad \bar{w}_i\equiv\frac{w_i}{\rho_e v_e},\quad c_T=16\int_0^\infty \rho v_x(H_e-H)\,r\,dr,$$

$$l_i=\left[\frac{1}{2\,\mathrm{Sc}_i^*}-\frac{u_0\,\mathrm{Sc}_i^*}{(\mathrm{Sc}_i^*+1)(2\mathrm{Sc}_i^*+1)}\right]^{-1}\left(\frac{\chi_{i0}}{\mathrm{Sc}_i^*}-\frac{u_0\chi_{i0}}{\mathrm{Sc}_i^*+1}-\frac{2\ln 2\alpha_i}{\Delta^2\rho_e v_e}\right),$$

$$J_i\equiv\int_0^1\left[\frac{\bar{w}_i}{\rho}-(1-u_0y)\,\bar{w}_{ie}\right]\frac{dy}{y},$$

$$A\equiv t_0\frac{2-a}{2\,\mathrm{Pr}^*}+\left(\sum_{i=1}^N\frac{\xi_{ie}}{M_i}\right)^{-1}\cdot\sum_{i=1}^N\frac{1}{M_i}\left\{\frac{2\chi_{i0}-l_i}{2\,\mathrm{Sc}_i^*}\right.$$
$$\left.+t_0\left[\frac{2(\chi_{i0}-l_i-a\chi_{i0})+3al_i}{2(\mathrm{Sc}_i^*+\mathrm{Pr}^*)}+\frac{a(\chi_{i0}-l_i)}{2\mathrm{Pr}^*+\mathrm{Sc}_i^*}+\frac{(1-a)\,l_i}{\mathrm{Pr}^*+2\mathrm{Sc}_i^*}\right]\right\}.$$

The constant c_T in (15) is determined from the initial conditions, and the quantities h_i are known functions of the temperature. The relationship (15) is used to define $a(x)$. The (N-2) equations (16) and (17) should be supplemented by the equalities

$$\sum_{i=1}^N\chi_{i0}=\sum_{i=1}^N l_i=0;\quad \sum_{i=1}^N Z_i\chi_{i0}/M_i=\sum_{i=1}^N Z_il_i/M_i=0. \tag{18}$$

The first two equalities follow from the integral $\sum_{i=1}^N \xi_i = 1$ of the system (4), and the second two follow from the quasineutrality condition.

In deriving the integral relations it was assumed that the stream tube equations

$$\rho_e v_e\frac{dv_e}{dx}=-\frac{dp}{dx},\quad H_e=H_\infty,\quad \rho_e v_e\frac{d\xi_{ie}}{dx}=w_{ie}$$

are valid on the outer boundary of the wake.

Let us note that if there is no pressure gradient ($v_e \equiv 1$), then the system (12), (13) has the integral

$$\Delta^2=c_x\ln 2\Big/\left[8u_0\left(1-\frac{u_0}{2}\right)\right]. \tag{19}$$

The initial conditions for the system (12)-(17) are obtained by using the approximations of the functions $v_H(r)$, $T_H(r)$, $\xi_{iH}(r)$ given in the initial section by the relationships (7)-(9).

Results of computations by the mesh method are presented in [1] for a reacting turbulent far wake behind a body moving at the velocity v_∞ = 7010 m/sec in a nitrogen atmosphere with T_∞ = 300°K, $p_\infty = 2\cdot 10^4$ N/m². The wake was computed by using an integral method under the same conditions as in [1], in order to compare with these exact numerical results.

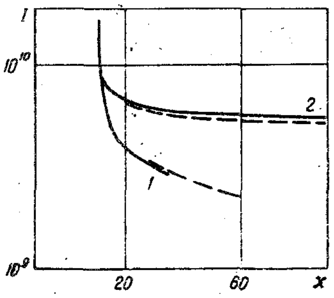

Fig. 4. Integral characteristic of the electron concentration.

A mixture of five components N, N^+, N_2^+, e, N_2 (i = 1, 2, 3, 4, 5, respectively) which took part in eight chemical reactions

$$e + N_2^+ = N + N, \quad N_2 + N_2 = 2N + N_2, \quad N + e + N^+ = N + N,$$
$$N_2 + e + N^+ = N_2 + N, \quad N + N_2^+ = N_2 + N^+, \quad N + e + N_2^+ = N + N_2,$$
$$N_2 + e + N_2^+ = N_2 + N_2, \quad N_2 + N = 3N$$

was considered. The forward reaction constants were taken from [1], while the reverse constants were computed by using the equilibrium constants according to data in [2].

It is assumed that the conditions on the outer wake boundary are the same as at infinity so that $v_e \equiv T_e \equiv \xi_{5e} = 1$, $\xi_{ie} = 0$, i = 1, 2, 3, 4.

The velocity and temperature profiles in the initial section are represented in Fig. 1a and b. It follows from Fig. 1a that in the initial section

$$1 - v_x(\delta_1) = 0.074\, u_0.$$

This relationship is taken as the definition of the wake radius δ_1.

It is assumed that chemical equilibrium holds in the initial section. In conformity with this assumption, the profiles of all the components were determined by means of the temperature and pressure by using tables [3]. The values of χ_{iH} and l_{iH} were determined by using approximations of the profiles constructed in such a manner by the relationship (9).

It is assumed that $Pr_T = Pr^* = Sc_{Ti} = Sc_i^* = 1$ in the computation. Molecular effects were not taken into account. The system of ordinary differential equations (12)-(14), (16), (17), where $a(x)$ is given implicitly by (15) and Δ is defined by (19), is integrated by the Runge-Kutta method on the BÉSM-3 electronic digital computer by using the relationship (18).

Presented in Figs. 2-4 is a comparison of the computed data obtained by using the integral relations method (dashes) and the mesh method (solid lines). The numbers 1, 2 denote results obtained by using the expressions

$$\mu_T = k(1 - \rho_0 v_0)\delta_1, \quad \mu_T = k u_0 \delta_1 \quad (k = 0.02)$$

respectively, for the turbulent viscosity. Presented in Fig. 2 is the wake radius, in Fig. 3 the dimensional axial temperature TT^0_∞K, and in Fig. 4 the dimensional integral electron concentration characteristic

$$I = \int_0^{\delta_1} n_4 dr, \quad 1\ \text{cm}^{-2}.$$

The agreement between the computed data obtained by using the approximate and exact numerical methods is completely satisfactory. Meanwhile, the approximate method has obvious advantages, the possibility of using standard programs in the computation in an electronic computer, and the savings in machine time.

NOTATION

x, r(d)†	are the longitudinal and radial coordinates;
v_x, v_r (v_∞)	are the velocity vector components on the x and r axes;
ρ (ρ_∞)	is the density;
p ($\rho_\infty v_\infty^2$)	is the pressure;
T (T_∞)	is the temperature;
H (v_∞^2)	is the total heat content;
h_i ($c_{p\infty} T_\infty$)	is the specific heat content;
μ_T, μ ($\rho_\infty v_\infty d$)	are the turbulent and laminar coefficients of viscosity;
Pr, Pr_T, Sc_i, Sc_{Ti}	are the physical and turbulent Prandtl and Schmidt numbers;
ξ_i	is the relative mass concentration of the i-th component;
J_{ri} ($\rho_\infty v_\infty$)	is the radial component of the diffusion vector;
w_i ($\rho_\infty v_\infty$/d)	is the mass rate of formation of the i-th component;

†In the parentheses the dimensional divisor for the given dimensionless quantity is shown.

M_∞	is the unperturbed stream Mach number;
$\varkappa$	is the adiabatic index;
M_i	is the molecular weight;
u_0, t, χ_i	see (7)-(9);
Pr^*, Sc_i^*	are the constants in the profiles (8), (9);
a, l_i	are the parameters of the profiles (8) and (9);
η, Δ	are the transformed radial coordinate and semi-radius of the wake (see (10));
$c_p \equiv \Sigma\xi_i \cdot \partial h_i/\partial T$;	
N	is the number of mixture components;
Z_i	is the charge number;
c_x	is the drag coefficient;
δ_1	is the wake radius;
k	is an empirical constant;
n_4	is the numberical electron density.

Subscripts

∞	denotes the undisturbed stream;
e	denotes the outer wake boundary;
0	denotes the wake axis;
H	denotes the initial section;
i	denotes the mixture component.

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